

2003 AP calculus BC EXAM 答案

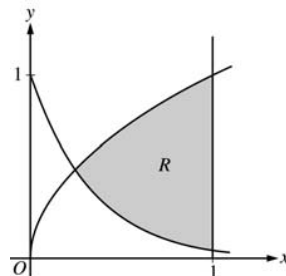
1-5	ECADC
6-10	CCBAC
11-15	DEAEB
16-20	CACDD
21-25	BEBDD
26-28	DED
76-80	EACDA
81-85	AAECA
86-90	CCCDB
91-92	BC

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Question 1

Let R be the shaded region bounded by the graphs of $y = \sqrt{x}$ and $y = e^{-3x}$ and the vertical line $x = 1$, as shown in the figure above.

- (a) Find the area of R .
- (b) Find the volume of the solid generated when R is revolved about the horizontal line $y = 1$.
- (c) The region R is the base of a solid. For this solid, each cross section perpendicular to the x -axis is a rectangle whose height is 5 times the length of its base in region R . Find the volume of this solid.



Point of intersection

$$e^{-3x} = \sqrt{x} \text{ at } (T, S) = (0.238734, 0.488604)$$

$$\begin{aligned} \text{(a) Area} &= \int_T^1 (\sqrt{x} - e^{-3x}) dx \\ &= 0.442 \text{ or } 0.443 \end{aligned}$$

$$\begin{aligned} \text{(b) Volume} &= \pi \int_T^1 ((1 - e^{-3x})^2 - (1 - \sqrt{x})^2) dx \\ &= 0.453\pi \text{ or } 1.423 \text{ or } 1.424 \end{aligned}$$

$$\begin{aligned} \text{(c) Length} &= \sqrt{x} - e^{-3x} \\ \text{Height} &= 5(\sqrt{x} - e^{-3x}) \end{aligned}$$

$$\text{Volume} = \int_T^1 5(\sqrt{x} - e^{-3x})^2 dx = 1.554$$

1: Correct limits in an integral in
(a), (b), or (c)

2 : $\left\{ \begin{array}{l} 1 : \text{integrand} \\ 1 : \text{answer} \end{array} \right.$

3 : $\left\{ \begin{array}{l} 2 : \text{integrand} \\ < -1 > \text{ reversal} \\ < -1 > \text{ error with constant} \\ < -1 > \text{ omits 1 in one radius} \\ < -2 > \text{ other errors} \\ 1 : \text{answer} \end{array} \right.$

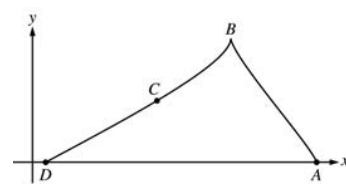
3 : $\left\{ \begin{array}{l} 2 : \text{integrand} \\ < -1 > \text{ incorrect but has} \\ & \sqrt{x} - e^{-3x} \\ & \text{as a factor} \\ 1 : \text{answer} \end{array} \right.$

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Question 2

A particle starts at point A on the positive x -axis at time $t = 0$ and travels along the curve from A to B to C to D , as shown above. The coordinates of the particle's position $(x(t), y(t))$ are differentiable functions of t , where

$$x'(t) = \frac{dx}{dt} = -9\cos\left(\frac{\pi t}{6}\right)\sin\left(\frac{\pi\sqrt{t+1}}{2}\right) \text{ and } y'(t) = \frac{dy}{dt} \text{ is not explicitly given.}$$



At time $t = 9$, the particle reaches its final position at point D on the positive x -axis.

- (a) At point C , is $\frac{dy}{dt}$ positive? At point C , is $\frac{dx}{dt}$ positive? Give a reason for each answer.
- (b) The slope of the curve is undefined at point B . At what time t is the particle at point B ?
- (c) The line tangent to the curve at the point $(x(8), y(8))$ has equation $y = \frac{5}{9}x - 2$. Find the velocity vector and the speed of the particle at this point.
- (d) How far apart are points A and D , the initial and final positions, respectively, of the particle?

- (a) At point C , $\frac{dy}{dt}$ is not positive because $y(t)$ is decreasing along the arc BD as t increases.
 At point C , $\frac{dx}{dt}$ is not positive because $x(t)$ is decreasing along the arc BD as t increases.

- (b) $\frac{dx}{dt} = 0$; $\cos\left(\frac{\pi t}{6}\right) = 0$ or $\sin\left(\frac{\pi\sqrt{t+1}}{2}\right) = 0$
 $\frac{\pi t}{6} = \frac{\pi}{2}$ or $\frac{\pi\sqrt{t+1}}{2} = \pi$; $t = 3$ for both.
 Particle is at point B at $t = 3$.

- (c) $x'(8) = -9\cos\left(\frac{4\pi}{3}\right)\sin\left(\frac{3\pi}{2}\right) = -\frac{9}{2}$
 $\frac{y'(8)}{x'(8)} = \frac{dy}{dx} = \frac{5}{9}$
 $y'(8) = \frac{5}{9}x'(8) = -\frac{5}{2}$
 The velocity vector is $\langle -4.5, -2.5 \rangle$.
 Speed $= \sqrt{4.5^2 + 2.5^2} = 5.147$ or 5.148

- (d) $x(9) - x(0) = \int_0^9 x'(t) dt$
 $= -39.255$
 The initial and final positions are 39.255 apart.

$$2 : \begin{cases} 1 : \frac{dy}{dt} \text{ not positive with reason} \\ 1 : \frac{dx}{dt} \text{ not positive with reason} \end{cases}$$

$$2 : \begin{cases} 1 : \text{sets } \frac{dx}{dt} = 0 \\ 1 : t = 3 \end{cases}$$

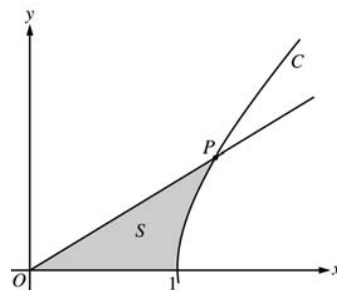
$$3 : \begin{cases} 1 : x'(8) \\ 1 : y'(8) \\ 1 : \text{speed} \end{cases}$$

$$2 : \begin{cases} 1 : \text{integral} \\ 1 : \text{answer} \end{cases}$$

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Question 3

The figure above shows the graphs of the line $x = \frac{5}{3}y$ and the curve C given by $x = \sqrt{1 + y^2}$. Let S be the shaded region bounded by the two graphs and the x -axis. The line and the curve intersect at point P .



- (a) Find the coordinates of point P and the value of $\frac{dx}{dy}$ for curve C at point P .
- (b) Set up and evaluate an integral expression with respect to y that gives the area of S .
- (c) Curve C is a part of the curve $x^2 - y^2 = 1$. Show that $x^2 - y^2 = 1$ can be written as the polar equation $r^2 = \frac{1}{\cos^2 \theta - \sin^2 \theta}$.
- (d) Use the polar equation given in part (c) to set up an integral expression with respect to the polar angle θ that represents the area of S .

- (a) At P , $\frac{5}{3}y = \sqrt{1 + y^2}$, so $y = \frac{3}{4}$.
Since $x = \frac{5}{3}y$, $x = \frac{5}{4}$.

$$\frac{dx}{dy} = \frac{y}{\sqrt{1 + y^2}} = \frac{y}{x}. \text{ At } P, \frac{dx}{dy} = \frac{3/4}{5/4} = \frac{3}{5}.$$

- (b) Area = $\int_0^{3/4} \left(\sqrt{1 + y^2} - \frac{5}{3}y \right) dy$
= 0.346 or 0.347

- (c) $x = r \cos \theta$; $y = r \sin \theta$
 $x^2 - y^2 = 1 \Rightarrow r^2 \cos^2 \theta - r^2 \sin^2 \theta = 1$
 $r^2 = \frac{1}{\cos^2 \theta - \sin^2 \theta}$

- (d) Let β be the angle that segment OP makes with the x -axis. Then $\tan \beta = \frac{y}{x} = \frac{3/4}{5/4} = \frac{3}{5}$.

$$\begin{aligned} \text{Area} &= \int_0^{\tan^{-1}(3/5)} \frac{1}{2} r^2 d\theta \\ &= \frac{1}{2} \int_0^{\tan^{-1}(3/5)} \frac{1}{\cos^2 \theta - \sin^2 \theta} d\theta \end{aligned}$$

$$2 : \begin{cases} 1 : \text{coordinates of } P \\ 1 : \frac{dx}{dy} \text{ at } P \end{cases}$$

$$3 : \begin{cases} 1 : \text{limits} \\ 1 : \text{integrand} \\ 1 : \text{answer} \end{cases}$$

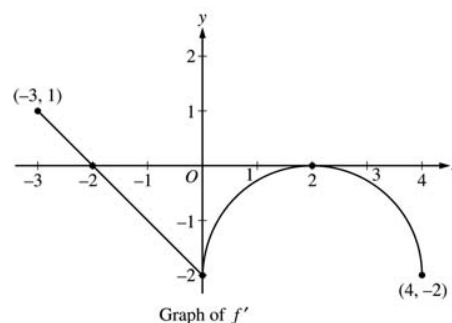
$$2 : \begin{cases} 1 : \text{substitutes } x = r \cos \theta \text{ and } y = r \sin \theta \text{ into } x^2 - y^2 = 1 \\ 1 : \text{isolates } r^2 \end{cases}$$

$$2 : \begin{cases} 1 : \text{limits} \\ 1 : \text{integrand and constant} \end{cases}$$

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Question 4

Let f be a function defined on the closed interval $-3 \leq x \leq 4$ with $f(0) = 3$. The graph of f' , the derivative of f , consists of one line segment and a semicircle, as shown above.



- (a) On what intervals, if any, is f increasing? Justify your answer.
- (b) Find the x -coordinate of each point of inflection of the graph of f on the open interval $-3 < x < 4$. Justify your answer.
- (c) Find an equation for the line tangent to the graph of f at the point $(0, 3)$.
- (d) Find $f(-3)$ and $f(4)$. Show the work that leads to your answers.

- (a) The function f is increasing on $[-3, -2]$ since $f' > 0$ for $-3 \leq x < -2$.

2 : $\begin{cases} 1 : \text{interval} \\ 1 : \text{reason} \end{cases}$

- (b) $x = 0$ and $x = 2$
 f' changes from decreasing to increasing at $x = 0$ and from increasing to decreasing at $x = 2$

2 : $\begin{cases} 1 : x = 0 \text{ and } x = 2 \text{ only} \\ 1 : \text{justification} \end{cases}$

- (c) $f'(0) = -2$
Tangent line is $y = -2x + 3$.

1 : equation

- (d) $f(0) - f(-3) = \int_{-3}^0 f'(t) dt$
 $= \frac{1}{2}(1)(1) - \frac{1}{2}(2)(2) = -\frac{3}{2}$

$$f(-3) = f(0) + \frac{3}{2} = \frac{9}{2}$$

$$f(4) - f(0) = \int_0^4 f'(t) dt$$

$$= -\left(8 - \frac{1}{2}(2)^2\pi\right) = -8 + 2\pi$$

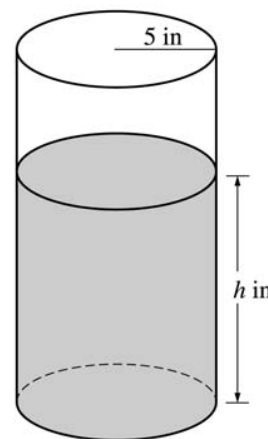
$$f(4) = f(0) - 8 + 2\pi = -5 + 2\pi$$

$\begin{cases} 1 : \pm\left(\frac{1}{2} - 2\right) \\ \text{(difference of areas of triangles)} \\ 1 : \text{answer for } f(-3) \text{ using FTC} \\ 4 : \begin{cases} 1 : \pm\left(8 - \frac{1}{2}(2)^2\pi\right) \\ \text{(area of rectangle} \\ \text{— area of semicircle)} \\ 1 : \text{answer for } f(4) \text{ using FTC} \end{cases} \end{cases}$

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Question 5

A coffeepot has the shape of a cylinder with radius 5 inches, as shown in the figure above. Let h be the depth of the coffee in the pot, measured in inches, where h is a function of time t , measured in seconds. The volume V of coffee in the pot is changing at the rate of $-5\pi\sqrt{h}$ cubic inches per second. (The volume V of a cylinder with radius r and height h is $V = \pi r^2 h$.)



- (a) Show that $\frac{dh}{dt} = -\frac{\sqrt{h}}{5}$.
- (b) Given that $h = 17$ at time $t = 0$, solve the differential equation $\frac{dh}{dt} = -\frac{\sqrt{h}}{5}$ for h as a function of t .
- (c) At what time t is the coffeepot empty?

(a) $V = 25\pi h$

$$\frac{dV}{dt} = 25\pi \frac{dh}{dt} = -5\pi\sqrt{h}$$

$$\frac{dh}{dt} = \frac{-5\pi\sqrt{h}}{25\pi} = -\frac{\sqrt{h}}{5}$$

$$3 : \begin{cases} 1 : \frac{dV}{dt} = -5\pi\sqrt{h} \\ 1 : \text{computes } \frac{dV}{dt} \\ 1 : \text{shows result} \end{cases}$$

(b) $\frac{dh}{dt} = -\frac{\sqrt{h}}{5}$

$$\frac{1}{\sqrt{h}} dh = -\frac{1}{5} dt$$

$$2\sqrt{h} = -\frac{1}{5}t + C$$

$$2\sqrt{17} = 0 + C$$

$$h = \left(-\frac{1}{10}t + \sqrt{17}\right)^2$$

$$5 : \begin{cases} 1 : \text{separates variables} \\ 1 : \text{antiderivatives} \\ 1 : \text{constant of integration} \\ 1 : \text{uses initial condition } h = 17 \\ \quad \text{when } t = 0 \\ 1 : \text{solves for } h \end{cases}$$

Note: max 2/5 [1-1-0-0-0] if no constant of integration

Note: 0/5 if no separation of variables

(c) $\left(-\frac{1}{10}t + \sqrt{17}\right)^2 = 0$

1 : answer

$$t = 10\sqrt{17}$$

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Question 6

The function f is defined by the power series

$$f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n+1)!} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \cdots + \frac{(-1)^n x^{2n}}{(2n+1)!} + \cdots$$

for all real numbers x .

- (a) Find $f'(0)$ and $f''(0)$. Determine whether f has a local maximum, a local minimum, or neither at $x = 0$. Give a reason for your answer.
- (b) Show that $1 - \frac{1}{3!}$ approximates $f(1)$ with error less than $\frac{1}{100}$.
- (c) Show that $y = f(x)$ is a solution to the differential equation $xy' + y = \cos x$.

- (a) $f'(0) = \text{coefficient of } x \text{ term} = 0$

$$f''(0) = 2 \text{ (coefficient of } x^2 \text{ term)} = 2\left(-\frac{1}{3!}\right) = -\frac{1}{3}$$

f has a local maximum at $x = 0$ because $f'(0) = 0$ and $f''(0) < 0$.

- (b) $f(1) = 1 - \frac{1}{3!} + \frac{1}{5!} - \frac{1}{7!} + \cdots + \frac{(-1)^n}{(2n+1)!} + \cdots$

This is an alternating series whose terms decrease in absolute value with limit 0. Thus, the error is less than the first omitted term, so $\left|f(1) - \left(1 - \frac{1}{3!}\right)\right| \leq \frac{1}{5!} = \frac{1}{120} < \frac{1}{100}$.

- (c) $y' = -\frac{2x}{3!} + \frac{4x^3}{5!} - \frac{6x^5}{7!} + \cdots + \frac{(-1)^n 2nx^{2n-1}}{(2n+1)!} + \cdots$

$$xy' = -\frac{2x^2}{3!} + \frac{4x^4}{5!} - \frac{6x^6}{7!} + \cdots + \frac{(-1)^n 2nx^{2n}}{(2n+1)!} + \cdots$$

$$xy' + y = 1 - \left(\frac{2}{3!} + \frac{1}{3!}\right)x^2 + \left(\frac{4}{5!} + \frac{1}{5!}\right)x^4 - \left(\frac{6}{7!} + \frac{1}{7!}\right)x^6 + \cdots$$

$$+ (-1)^n \left(\frac{2n}{(2n+1)!} + \frac{1}{(2n+1)!}\right)x^{2n} + \cdots$$

$$= 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \cdots + \frac{(-1)^n}{(2n)!}x^{2n} + \cdots$$

$$= \cos x$$

OR

$$xy = xf(x) = x - \frac{x^3}{3!} + \cdots + (-1)^n \frac{1}{(2n+1)!} x^{2n+1} + \cdots$$

$$= \sin x$$

$$xy' + y = (xy)' = (\sin x)' = \cos x$$

$$4 : \begin{cases} 1 : f'(0) \\ 1 : f''(0) \\ 1 : \text{critical point answer} \\ 1 : \text{reason} \end{cases}$$

$$1 : \text{error bound} < \frac{1}{100}$$

$$4 : \begin{cases} 1 : \text{series for } y' \\ 1 : \text{series for } xy' \\ 1 : \text{series for } xy' + y \\ 1 : \text{identifies series as } \cos x \end{cases}$$

OR

$$4 : \begin{cases} 1 : \text{series for } xf(x) \\ 1 : \text{identifies series as } \sin x \\ 1 : \text{handles } xy' + y \\ 1 : \text{makes connection} \end{cases}$$